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A Uniqueness Result for an Optimal Control Problem on a Diffusive Elliptic Volterra–Lotka Type Equation

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Uniqueness for an optimal control problem with logistic elliptic equation is shown providing that the quotient between the crowding effect and the benefit ratio is large enough. Also, a sub–super solution method is developed in order to approximate the optimal control.

Key Words: Optimal control; logistic elliptic equations; existence; uniqueness; lower-upper solutions.

1. INTRODUCTION

In this work we are going to study the profitability of a biological growing species whose growth in a bounded domain $\Omega \subset \mathbb{R}^N$ is modelling by the logistic elliptic equation

$$\begin{aligned} -\Delta u(x) &= u(x)\left[a(x) - f(x) - b(x)u(x)\right], && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega \end{aligned} \tag{1.1}$$

A previous equation described the steady-state solutions for the corresponding time dependent parabolic model. Here, u represents the concentration of the biological species, function a is the growth rate, b means the crowding effect, and f plays the role of control. A control f works by decreasing the growth rate in order to increase the quality of the production. Our control space will be

$$L^\infty_+(\Omega) = \{g \in L^\infty(\Omega) : 0 \le g \text{ a.e. in } \Omega\}.$$

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With the symbol Δ we denote the laplacian operator. Boundary conditions are of Dirichlet type. The biological interpretation of this fact is that the species cannot live on the boundary of domain Ω.

Later on (see Section 2) for each $f \in L^\infty_+(\Omega)$, we will define $u_{\Omega, a, b, f}$ as the nonnegative maximal solution of problem (1.1). This fact allows us to define the quadratic payoff functional $J_{K, M} : L^\infty_+(\Omega) \to \mathbb{R}$ as

$$J_{K, M}(f) = \int_\Omega \left[K u_{\Omega, a, b, f}(x) f(x) - M f^2(x) \right] dx,$$

which expresses the difference between profit and cost. Here, $K > 0$ denotes the sale prices of the species, and $M > 0$ is the cost of the control.

The above payoff functional is considered in [10], in the study of an optimal control problem with Neumann boundary conditions and space of controls $C_\delta = \{g \in L^\infty(\Omega) : 0 \leq g \leq \delta\}$ for a fixed $\delta > 0$. In the case $\delta < \inf_{x \in \Omega} a(x)$ the authors prove the existence of an optimal control in C_δ. Under suitable conditions, they describe the optimal control f by using the optimality system. Ideas and tools contained in this paper can be used there in order to obtain uniqueness and approximation results for the mentioned Neumann control problem (see Final Remarks).

The corresponding optimal control problem for periodic parabolic equations was studied in [6] where a result about uniqueness and approximation to the optimal control is proved when parameter M is sufficiently large.

Other kinds of cost functionals for biological control problems can be found in [8]. There, an optimal control problem for parabolic equations with logistic growth is considered. These authors characterize the unique optimal control when the time interval $(0, T)$, $T > 0$, is small enough.

Obviously, our interest is to maximize $J_{K, M}$. For convenience, we consider the equivalent functional

$$\frac{J_{K, M}(f)}{M} = J(f) = \int_\Omega \lambda u_{\Omega, a, b, f} f - f^2, \tag{1.2}$$

where $\lambda = \frac{K}{M}$ will be called the benefit-to-cost ratio.

For $(P_{\Omega, a, b, \lambda})$ we denote the problem of finding admissible controls, $f \in L^\infty_+(\Omega)$, such that

$$J(f) = \sup_{L^\infty_+(\Omega)} J(g).$$

Such control, f, will be called an optimal control.

This problem has been studied in [3, 4]. In [3] it was proved that, for parameter λ small enough, the optimal control problem $(P_{\Omega, a, b, \lambda})$ has a unique optimal control and by using the optimality system it is possible to construct an iterative method that approaches the unique optimal control.

On the other hand, in [4], by assuming other kinds of hypotheses involving the crowding effect of the species, the uniqueness and approximation to the optimal control was obtained. Essentially, the uniqueness for problem $(P_{\Omega,a,b,\lambda})$ is shown, when function b satisfies

$$\exists \tilde{N} \in [1,2) \qquad \text{such that} \quad \sup b \le \tilde{N} \inf b, \tag{1.3}$$

and $\inf b$ is sufficiently large. The above uniqueness result is not completely satisfactory. In particular, in the case where function b is a multiple of a given function b_0, $b = \gamma b_0$, the control problem $(P_{\Omega,a,b,\lambda})$ has a unique optimal control if γ is sufficiently large, although b_0 fails condition (1.3) (see [4]).

In this work we prove a more general uniqueness result. By defining $\sigma_1(q)$, for $q \in L^\infty(\Omega)$, to be the principal eigenvalue of the corresponding eigenvalues problem (see (2.4) below), in Section 3 we prove

THEOREM 1.1 (Main Theorem). *Let $C \in \mathbb{R}^+$, $C > 1$ be a fixed constant $a \in L^\infty(\Omega)$ such that $\sigma_1(-a) < 0$. $\exists \mu \equiv \mu(\Omega, a, C) > 0$ such that, if $b \in L^\infty(\Omega)$ satisfies conditions*

$$\inf b > 0, \ \frac{\sup b}{\inf b} \le C, \quad \text{and} \quad \frac{\inf b}{\lambda} \ge \mu,$$

then $(P_{\Omega,a,b,\lambda})$ admits a unique optimal control.

Also, we will show that by using the optimality system it is possible to construct a monotone scheme to approach the optimal control (see Theorem 3.11).

The paper is organized as follows.

In Section 2 we fix some notation and we summarize some preliminary results about the existence and uniqueness of a positive solution for the logistic Equation (1.1), the existence of a solution for the control problem $(P_{\Omega,a,b,\lambda})$, and the optimality system. Also, we prove a lemma which permits the inversion of operator $-\Delta + (-a + 2bu)$ when u is a positive solution of a logistic equation (see Lemma 2.7 later on). This lemma will be essential to obtain Theorem 1.1 and Theorem 3.11 mentioned above. Section 3 includes the proofs of these theorems.

2. NOTATION AND PRELIMINARY RESULTS

In this section we will show some previous results and we will fix the notation that will be used in the following. Given a function $e \in L^\infty(\Omega)$ we denote $\bar{e} = \operatorname{ess\,sup}_\Omega e$, and accordingly $\underline{e} = \operatorname{ess\,inf}_\Omega e$.

From now on we consider the Equation (1.1) under the hypotheses

[H1] Ω is a bounded and regular domain in $\mathbb{R}^N$ $a, b \in L^\infty(\Omega)$, with $\underline{b} > 0$ and $f \in L^\infty_+(\Omega)$.

In some results we will need in addition the hypothesis

[H2] $1 \leq \overline{\mathrm{b}}/\underline{\mathrm{b}} \leq C$ for $C \in \mathbb{R}^+$ fixed constant.

Now, we recall the main properties of the principal eigenvalue for a suitable eigenvalue problem. For each $q \in L^\infty(\Omega)$ we consider $\sigma_1(q)$ to be the principal eigenvalue of the following eigenvalue problem

$$\begin{aligned} -\Delta u(x) + q(x)u(x) &= \lambda u(x), \qquad && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega. \end{aligned} \tag{2.4}$$

It is known (see, for instance, [5]) that $\sigma_1(q)$ has algebraic multiplicity equal to one and satisfies the variational characterization

$$\sigma_1(q) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + \int_\Omega qu^2}{\int_\Omega u^2} \tag{2.5}$$

where $H_0^1(\Omega)$ is the usual Sobolev space.

Moreover, one can choose a unique associated eigenfunction, $\phi_1(q)$, such that $\phi_1(q) \in C^{1,\alpha}(\overline{\Omega})$ (the space of Hölderian functions), $\forall \alpha \in (0,1)$, $\phi_1(q)$ strictly positive in Ω, and $\|\phi_1(q)\|_{L^\infty(\Omega)} = 1$.

As a consequence of (2.5), it follows that $\sigma_1(q)$ has the following properties:

(i) If $q_1, q_2 \in L^\infty(\Omega)$, with $q_1(x) \leq q_2(x)$, a.e. in Ω, then $\sigma_1(q_1) \leq \sigma_1(q_2)$. Moreover, if the set $\{x \in \Omega : q_1(x) < q_2(x)\}$ has positive measure, then $\sigma_1(q_1) < \sigma_1(q_2)$.

(ii) $\forall q \in L^\infty(\Omega)$, $\forall M \in \mathbb{R}$, and $\sigma_1(q + M) = \sigma_1(q) + M$.

(iii) The map σ_1: $L^\infty(\Omega) \to \mathbb{R}$, $q \mapsto \sigma_1(q)$ is continuous.

(iv) Let $q_1, q_2 \in L^\infty(\Omega)$ and $t \in [0,1]$. Then

$$\sigma_1\big(tq_1 + (1-t)q_2\big) \geq t\sigma_1(q_1) + (1-t)\sigma_1(q_2). \tag{2.6}$$

We now point out some properties of the Schrödinger operator $(-\Delta + q)$ and of the nonnegative solutions of Equation (1.1) (for more details see [1–3, 7, 11, 12]).

For $u \in H^1(\Omega)$, the expression $u \geq 0$ in $\partial\Omega$ means $u^- \in H_0^1(\Omega)$.

LEMMA 2.2 ([3]). *Consider $q \in L^\infty(\Omega)$ satisfying $\sigma_1(q) > 0$. Then*

(i) *For each $f \in L^2(\Omega)$ the linear problem*

$$\begin{aligned} -\Delta u + q(x)u &= f, \qquad && in\ \Omega, \\ u &= 0, && on\ \partial\Omega, \end{aligned} \tag{2.7}$$

has a unique solution $u \in H_0^1(\Omega)$; *moreover, if* $f \in L^\infty(\Omega)$, $u \in W^{2,p}(\Omega)$, $\forall p \in (1, \infty)$.

(ii) *Let* $u \in H^1(\Omega)$ *satisfying* $u \geq 0$ *on* $\partial\Omega$ *and* $-\Delta u + qu \geq 0$, *in the weak sense, i.e.,* $\forall \phi \in H_0^1(\Omega)$, $\phi \geq 0$,

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega q u \phi \geq 0,$$
$$u \geq 0, \qquad on\ \partial\Omega.$$

Then $u \geq 0$ *in* Ω.

(iii) *Consider* $p_1, p_2 \in L^\infty(\Omega)$, $p_1 \geq p_2$ *in* Ω *with* $\sigma_1(p_2) > 0$. *Let* f *be in* $L^2(\Omega)$ *such that* $f \geq 0$ *in* Ω. *Denote by* ω_{p_i}, *the corresponding solution of problem*

$$\begin{aligned} -\Delta u + p_i(x) u &= f, &\qquad in\ \Omega,\\ u &= 0, &\qquad on\ \partial\Omega, \end{aligned}$$

for $i = 1, 2$.

Then

$$\omega_{p_1}(x) \leq \omega_{p_2}(x), \qquad a.e.\ in\ \Omega.$$

By using techniques of sub- and supersolutions and the variational characterization of $\sigma_1(-a)$ (see (2.5)), one may prove [1, 7] that (1.1) has a (weak) nonnegative and nontrivial solution iff $\sigma_1(-a+f) < 0$. In this case, there exists a unique nonnegative and nontrivial solution of (1.1), denoted by u, satisfying the estimates

$$\frac{-\sigma_1(-a+f)}{\bar{b}} \phi_1(-a+f)(x) \leq u(x) \leq \frac{\bar{a} - \underline{f}}{\underline{b}}, \qquad \forall x \in \Omega. \quad (2.8)$$

Moreover, this solution u is stable, considered as an equilibrium solution of the corresponding time dependent evolution problem [12].

For each $f \in L_+^\infty(\Omega)$ (and in the same way, for each $f \in L^\infty(\Omega)$), the above considerations permit us to define $u_{\Omega,a,b,f}$ (or $u_{b,f}$, when there is not ambiguity), as the maximal nonnegative solution of Eq. (1.1). Then, $u_{\Omega,a,b,f} \equiv u > 0$ in Ω iff $\sigma_1(-a+f) < 0$ and $u_{\Omega,a,b,f} \equiv 0$ iff $\sigma_1(-a+f) \geq 0$.

Observe that the upper estimate for the maximal nonnegative solution of (1.1), $u_{\Omega,a,b,f} \leq \max\{0, \bar{a}/\underline{b}\}$, is uniform for $f \in L_+^\infty(\Omega)$.

The main properties of solution $u_{\Omega,a,b,f}$ used through this work are shown in the following proposition

PROPOSITION 2.3 (Main properties of $u_{\Omega,a,b,f}$). *Under hypothesis* [H1], *we have*

(1) *The map $L^\infty(\Omega) \to C^1(\overline{\Omega})$, $f \mapsto u_{b,f}$ is continuous and decreasing in the following sense: if $f, g \in L^\infty(\Omega)$ such that $f \le g$, a.e. in Ω, then $u_{b,f}(x) \ge u_{b,g}(x)$, $\forall x \in \Omega$. Analogously, the map $b \mapsto u_{b,f}$ is decreasing; i.e., if b_1, b_2 satisfy* [H1] *and $b_1 \ge b_2$ a.e. in Ω, then $u_{b_1,f} \le u_{b_2,f}$ in Ω.*

(2) *The above function $f \mapsto u_{b,f}$ defined from the open $A = \{h \in L^\infty(\Omega): \sigma_1(-a+h) < 0\}$ to $W^{1,p}(\Omega)$, $p > N$, is C^∞.*

Proof. The proof of part 1 can be found in [2].

To show the second part we use the function implicit theorem. Let $F\colon\ A \subset L^\infty(\Omega) \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, $F(f,u) = u + \mathscr{K}(u(a-f) - bu^2)$, where $\mathscr{K}$ is the linear and compact operator defined by formula $\mathscr{K} \equiv (-\Delta)^{-1}$. Let $f_0 \in A$; it is clear that (f_0, u_{b,f_0}) is a solution of the equation $F(f,u) = 0$. This fact implies, in particular, that $\sigma_1(-a + f_0 + bu_{b,f_0}) = 0$ (observe that 0 is an eigenvalue with "positive" eigenfunction u_{b,f_0} for the corresponding eigenvalues problem). By using the monotonicity of $\sigma_1(\cdot)$, the positivity of u_{b,f_0}, and (i) of Lemma 2.2 we obtain that $\forall g \in L^p(\Omega)$ and there exists a unique $v \in W_0^{1,p}(\Omega)$ such that

$$-\Delta v + \left(-a + f_0 + 2bu_{b,f_0}\right)v = g.$$

Now, the Fredholm alternative theorem assures that $\frac{\partial F}{\partial u}(f_0, u_{b,f_0})\colon W_0^{1,p} \to W_0^{1,p}$, $v \mapsto \frac{\partial F}{\partial u}(f_0, u_{b,f_0})(v) = v - \mathscr{K}((a-f_0)v - 2bu_{b,f_0}v)$ is a linear isomorphism. Thus, there are two open neighborhoods $O_1 \subset L^\infty(\Omega)$, $O_2 \subset W_0^{1,p}(\Omega)$ of f_0, u_{b,f_0}, respectively, and there exists a unique C^∞-differentiable map $T\colon O_1 \to O_2$ such that $T(f_0) = u_{b,f_0}$ and $F(f, T(f)) = 0$. The implicit function theorem allows us to calculate the expression of the differential. In fact, $\forall g \in L^\infty(\Omega)$, the differential $DT_f(g)$ is the unique solution of the linear problem

$$\begin{aligned} -\Delta \xi_{f,g} + \left[-a + f + 2bu_{b,f}\right]\xi_{f,g} &= -gu_{b,f}, && \text{in } \Omega \\ \xi_{f,g} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{2.9}$$

This concludes the proof. ∎

The existence of a solution for the problem $(P_{\Omega,a,b,\lambda})$, i.e., the existence of a control $f \in L_+^\infty(\Omega)$ which maximizes the benefit-cost functional J, is shown in [3]. There it is proved that if $\bar{a} > 0$ and if f is an optimal control, then we may assume that

$$f \le \lambda \frac{\bar{a}}{\underline{b}}, \qquad \text{a.e. in } \Omega. \tag{2.10}$$

(Observe that $\bar{a} \le 0$ implies $\sigma_1(-a+f) \ge 0$ and then necessarily the optimal control $f \equiv 0$.) Also, the positivity of benefit is characterized in terms of Ω and function a. Precisely, in [3] is proved that, under hypothesis [H1],

$$\sup_{g \in L^\infty_+(\Omega)} J(g) > 0 \quad \Leftrightarrow \quad \sigma_1(-a) < 0.$$

This last result justifies the hypothesis $\sigma_1(-a) < 0$ in many of the following results.

Lemma 2.4 ([3, Lemma 3.2]). *Assume* [H1] *and* $\sigma_1(-a) < 0$. *If* $f \in L^\infty_+(\Omega)$ *is an optimal control, then*

$$f = \frac{\lambda}{2} u_{b,f} \left(1 - p_{b,f}\right)^+, \qquad a.e. \text{ in } \Omega,$$

where $p_{b,f}$ *is the unique solution of the linear* (*adjoint*) *problem*

$$\begin{aligned} -\Delta p_{b,f} + \left(-a + f + 2bu_{b,f}\right) p_{b,f} &= f, \qquad \text{in } \Omega, \\ p_{b,f} &= 0, \qquad \text{on } \partial\Omega \end{aligned} \tag{2.11}$$

(*Observe that* $\sigma_1(-a+f+2bu_{b,f}) > \sigma_1(-a+f+bu_{b,f}) = 0$.)

The previous lemma and Proposition 2.3 (part 2) are essential to have the optimality system

Corollary 2.5 (optimality system, [3]). *Under hypothesis* [H1] *and* $\sigma_1(-a) < 0$, *if* $f \in L^\infty_+(\Omega)$ *is an optimal control, then*

$$f = \frac{\lambda}{2} u (1-p)^+, \qquad \text{in } \Omega, \tag{2.12}$$

where (u, p) *is a solution of the optimality system*

$$\begin{aligned} -\Delta u &= u\left(a - \left[b + \frac{\lambda}{2}(1-p)^+\right]u\right), \qquad \text{in } \Omega, \\ -\Delta p + p(-a + 2bu) &= \frac{\lambda}{2} u \left[(1-p)^+\right]^2, \qquad \text{in } \Omega, \\ u = p &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.13}$$

with conditions

$$0 \leq p, \quad u > 0, \qquad a.e. \text{ in } \Omega. \tag{2.14}$$

Remark 2.6. The optimality system (2.13) and (2.14), in particular, shows that if $f \in L^{\infty}_{+}(\Omega)$ is an optimal control, then f is Lipschitz continuous (the extra regularity of the optimal control comes through the characterization of the optimal control and the regularity of u and p). Moreover, in the case $p_{b,f}(x) < 1$, $x \in \Omega$, we get $(1 - p_{b,f})^{+} = (1 - p_{b,f})$ in Ω and consequently more regularity for the optimal control (actually $f \in W^{2,q}(\Omega)$, $\forall q \in (1, \infty)$). Also, in this case, we obtain the positivity of the optimal control ($f(x) > 0$, $x \in \Omega$).

In the next lemmas, we will show that, under hypotheses [H1] and [H2], it is possible to obtain a positive lower bound for the expression $\sigma_1(-a + 2bu)$ and an upper bound for function p, provided that $\sigma_1(-a) < 0$ and (u, p) corresponds to a solution of optimality system (2.13) and (2.14). The interest in these results will be to prove in Section 3 the uniqueness of the optimal control, for a more general setting than Corollaries 4.3 and 4.4 and Theorem 3.5 shown in [3, 4], respectively.

LEMMA 2.7. *Suppose* [H1], [H2], $\sigma_1(-a) < 0$, *and* (u, p) *is a solution of* (2.13) *and* (2.14). *Then* $\exists \tau > 0$ *such that*

$$\sigma_1(-a + 2bu) \geq \tau \qquad \text{uniformly for } \frac{\lambda}{\underline{b}} \in]0, 1]. \tag{2.15}$$

Proof. Consider w as the maximal nonnegative solution of the problem

$$\begin{aligned} -\Delta w &= w\left[a(x) - \left(\frac{\lambda}{2} + b(x)\right)w\right], && \text{in } \Omega, \\ w &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{2.16}$$

Observe that $\frac{\lambda}{2}(1 - p)^{+} \leq \frac{\lambda}{2}$ and the monotonocity of $b \mapsto u_{b,f}$ (Proposition 2.3) imply

$$u \equiv u_{b+(\lambda/2)(1-p)^{+},0} \geq w \equiv u_{b+(\lambda/2),0}. \tag{2.17}$$

If b is a function verifying [H1] and [H2] and $f \in L^{\infty}_{+}(\Omega)$ is an admissible control, again, Proposition 2.3 implies

$$bu_{b,f} \geq bu_{\overline{b},f} \geq \frac{\underline{b}}{\overline{b}}\overline{b}u_{\overline{b},f} \geq \frac{1}{C}\overline{b}u_{\overline{b},f} = \frac{1}{C}u_{1,f}. \tag{2.18}$$

So, we have obtained a lower bound for $bu_{b,f}$ uniformly with respect to b provided that $u_{b,f}$ is the corresponding associated solution. We will use this fact to conclude the proof. In fact, taking into account the properties of $\sigma_1(\cdot)$ and previous arguments we have

$$
\begin{aligned}
\sigma_1(-a + 2bu) &\geq \sigma_1(-a + 2bw) \\
&= \sigma_1\left(-a + \left(b + \frac{\lambda}{2}\right)w + \left(b - \frac{\lambda}{2}\right)w\right) \\
&\geq \sigma_1\left(-a + \left(b + \frac{\lambda}{2}\right)u_{b+(\lambda/2),0} + \left(b - \frac{\lambda}{2}\right)u_{\bar{b}+(\lambda/2),0}\right).
\end{aligned}
$$

Now, denoting $-a + (b + (\lambda/2))\, w$ as z (recall that $w = u_{b+(\lambda/2),0}$ and $\sigma_1(z) = 0$), the above inequality should be

$$
\begin{aligned}
\sigma_1(-a + 2bu) &\geq \sigma_1\left(z + \left(b - \frac{\lambda}{2}\right)u_{\bar{b}+(\lambda/2),0}\right) \\
&\geq \sigma_1\left(z + \left(\frac{2\underline{b} - \lambda}{2\bar{b} + \lambda}\right)\left(\bar{b} + \frac{\lambda}{2}\right)u_{\bar{b}+(\lambda/2),0}\right) \\
&\geq \sigma_1\left(z + \left(\frac{\underline{b}}{3\bar{b}}\right)u_{1,0}\right) \\
&\geq \sigma_1\left(z + \left(\frac{1}{3C}\right)u_{1,0}\right).
\end{aligned}
$$

Fix $\theta \in (0,1)$; the concavity of $\sigma_1(\cdot)$ and $\sigma_1(z) = 0$ implies

$$
\begin{aligned}
&\sigma_1\left(z + \left(\frac{1}{3C}\right)u_{1,0}\right) \\
&\qquad = \sigma_1\left(\theta z + (1-\theta)\left[z + \left(\frac{1}{3C(1-\theta)}\right)u_{1,0}\right]\right) \\
&\qquad \geq (1-\theta)\,\sigma_1\left(z + \left(\frac{1}{3C(1-\theta)a}\right)u_{1,0}\right) \\
&\quad \cdots \geq (1-\theta)^n\,\sigma_1\left(z + \left(\frac{1}{3C(1-\theta)^n}\right)u_{1,0}\right),
\end{aligned}
$$

for each $n \in \mathbb{N}$. We need the lower estimate for $\sigma_1(z + (1/3C)u_{1,0})$ to be uniform with respect to function b. To see it, take some $\tilde{n} \in \mathbb{N}$ such that

$1/(3C(-\theta)^{\tilde{n}}) > 1$, then

$$\sigma_1\left(z + \left(\frac{1}{3C(1-\theta)^{\tilde{n}}}\right)u_{1,0}\right)$$
$$= \sigma_1\left(-a + u_{1,0} + \left(\frac{1}{3C(1-\theta)^{\tilde{n}}} - 1\right)u_{1,0} + \left(b + \frac{\lambda}{2}\right)w\right)$$
$$> \sigma_1\left(-a + u_{1,0} + \left(\frac{1}{3C(1-\theta)^{\tilde{n}}}\right)u_{1,0}\right) \equiv \tau > 0.$$

∎

The previous lemma says, in particular, that the operator $-\Delta + (-a + 2bu)$ is invertible; that is, for any $g \in L^2(\Omega)$ there exists a unique $v \in H_0^1(\Omega)$ solution of the problem

$$-\Delta v + (-a + 2bu)v = g.$$

where u corresponds with some solution (u, p) of the optimality system (2.13) and (2.14).

LEMMA 2.8 (estimates for the adjoint problem). *Assume* [H1], [H2], *and* $\sigma_1(-a) < 0$. *Let* (u, p) *be a solution of* (2.13)–(2.14). *Then, there exists a constant* $R > 0$ *such that*

$$0 \le p \le \frac{\lambda}{\underline{b}} R, \qquad \text{in } \Omega, \tag{2.19}$$

uniformly for $\frac{\lambda}{\underline{b}} \in]0, 1]$.

Proof. The proof is a consequence of Lemmas 2.2 and 2.7. In fact, let us consider $\mathscr{Q}$ to be the unique solution of problem

$$\begin{aligned} -\Delta\mathscr{Q} + (-a + 2bw)\mathscr{Q} &= \bar{a}, && \text{in } \Omega, \\ \mathscr{Q} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{2.20}$$

Thus, $\lambda\mathscr{Q}/\underline{b}$ is a solution of

$$\begin{aligned} -\Delta\frac{\lambda\mathscr{Q}}{\underline{b}} + (-a + 2bw)\frac{\lambda\mathscr{Q}}{\underline{b}} &= \frac{\lambda\bar{a}}{\underline{b}}, && \text{in } \Omega, \\ \frac{\lambda\mathscr{Q}}{\underline{b}} &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.21}$$

and applying Lemma 2.2 and (2.17) we get

$$0 \leq p \leq \frac{\lambda \mathscr{Q}}{\underline{b}}, \qquad \text{in } \Omega. \tag{2.22}$$

Now, multiplying by $\mathscr{Q}$ in (2.20), integrating on Ω, and taking into account (2.5) and Lemma 2.7, we obtain

$$\tau \int_\Omega \mathscr{Q}^2 \leq \sigma_1(-a + 2bw) \int_\Omega \mathscr{Q}^2 \leq \int_\Omega |\nabla \mathscr{Q}|^2 + (-a + 2bw)\mathscr{Q}^2 = \int_\Omega \bar{a} \mathscr{Q}.$$

The Holder inequality gives

$$\tau \|\mathscr{Q}\|_2^2 \leq \bar{a} \|\mathscr{Q}\|_2 \, |\Omega|_2 \tag{2.23}$$

or, equivalently,

$$\tau \|\mathscr{Q}\|_2 \leq \bar{a} \, |\Omega|_2. \tag{2.24}$$

Standard elliptic regularity arguments do the rest. ∎

3. UNIQUENESS AND APPROXIMATION TO THE OPTIMAL CONTROL

In this section, we prove the uniqueness result for the problem $(P_{\Omega, a, b, \lambda})$ mentioned in the Introduction (Theorem 1.1). Also, by using an iterative scheme with an appropriate system of sub- and supersolution for the optimality system (2.13) and (2.14), we can approximate to the unique optimal control. The new information about the behavior of $\sigma_1(-a + 2bu)$ and p, for a solution (u, p) of (2.13) and (2.14), contained in Lemmas 2.7 and 2.8, permits us an improvement and unification of known results about this control problem (see [3, 4]).

Proof of Theorem 1.1. By following the notation introduced in Section 2, we are going to prove that if

$$\frac{\underline{b}^2}{\lambda^2} \geq \Gamma \equiv \max\left\{ \frac{\bar{a}^2\left(2CR + \frac{1}{2}\right)}{2\tau^2}, R^2, 1 \right\}, \tag{3.25}$$

where τ and R are defined in Lemmas 2.7 and 2.8, respectively, then there exists a unique solution (u, p) of the optimality system (2.13) and (2.14). In particular, the control problem $(P_{\Omega, a, b, \lambda})$ has an unique optimal control.

Let us denote $\alpha = \lambda/\tau(2CR + \frac{1}{2})$. So, (u, p) is a solution of (2.13) and (2.14), if and only if $(u, r) = (u, \frac{p}{\alpha})$ is a solution of system

$$\begin{aligned}
-\Delta u - au + bu^2 + \frac{\lambda}{2}(1 - r\alpha)u^2 &= 0, \qquad \text{in } \Omega,\\
-\Delta r + r(-a + 2bu) - \frac{\lambda}{2\alpha}u(1 - r\alpha)^2 &= 0, \qquad \text{in } \Omega,\\
u = r &= 0, \qquad \text{on } \partial\Omega.
\end{aligned} \tag{3.26}$$

with conditions

$$0 \le r(x) \le \frac{\lambda R}{\underline{b}\alpha}\left(\le \frac{1}{\alpha}\right), w(x) \le u(x) \le \frac{\bar{a}}{\underline{b}}, \qquad \text{a.e. in } \Omega. \tag{3.27}$$

We will prove that system (3.26) and (3.27) has only one solution. In fact, let (v, s) be another solution of the mentioned system. By subtracting we obtain

$$\begin{aligned}
0 = {} & -\Delta(u - v) - a(u - v) + b(u^2 - v^2)\\
& + \frac{\lambda}{2}(u^2 - v^2) - \frac{\lambda\alpha}{2}(u^2 r - v^2 s)
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
0 = {} & -\Delta(r - s) - a(r - s) + 2b(ur - vs)\\
& - \frac{\lambda\alpha}{2}\left(u\left(\frac{1}{\alpha} - r\right)^2 - v\left(\frac{1}{\alpha} - s\right)^2\right).
\end{aligned} \tag{3.29}$$

Now, by multiplying (3.26) by $(u - v)$ and (3.27) by $(r - s)$, integrating on Ω, and adding both expressions, we have

$$\begin{aligned}
0 = {} & \int_\Omega \bigg[|\nabla(u - v)|^2 - a(u - v)^2 + b(u + v)(u - v)^2\\
& \qquad + \frac{\lambda}{2}(u + v)(u - v)^2(1 - \alpha s) - \frac{\lambda\alpha}{2}u^2(u - v)(r - s)\bigg]\\
& + \int_\Omega \bigg[|\nabla(r - s)|^2 - a(r - s)^2 + 2bu(r - s)^2\\
& \qquad + (u - v)(r - s)\left(2bs - \frac{\lambda\alpha}{2}\left(\frac{1}{\alpha} - s\right)^2\right)\\
& \qquad + \frac{\lambda\alpha}{2}u\left(\frac{2}{\alpha} - (r + s)\right)(r - s)^2\bigg].
\end{aligned} \tag{3.30}$$

Now, taking into account that the term

$$\frac{\lambda}{2}(u+v)(u-v)^2(1-\alpha s) + \frac{\lambda\alpha}{2}u\left(\frac{2}{\alpha}-(r+s)\right)(r-s)^2$$

is nonnegative (really is strictly positive if $r(x) \neq s(x)$ in a subset of positive measure of Ω), then, the inequalities

$$\sigma_1(-a+(u+v)b) \geq \tau$$

$$\sigma_1(-a+2bu) \geq \tau$$

given by Lemma 2.7 and the variational characterization for $\sigma_1(\cdot)$ (recall (2.5)), imply

$$0 \geq \int_\Omega \Bigg[\tau(u-v)^2 + \tau(r-s)^2 + (u-v)(r-s)\left(2bs - \frac{\lambda\alpha}{2}\left(\frac{1}{\alpha}-s\right)^2 - \frac{\lambda\alpha}{2}u^2\right)\Bigg], \quad (3.31)$$

with strict inequality if $r(x) \neq s(x)$ is a subset of Ω with positive measure. The choice of α, hypothesis (3.25) and (3.27), gives us that

$$\left|2bs - \frac{\lambda\alpha}{2}\left(\frac{1}{\alpha}-s\right)^2\right| \leq \frac{2\bar{b}\lambda R}{\underline{b}\alpha} + \frac{\lambda}{2\alpha} = \tau \quad (3.32)$$

and

$$\left|\frac{\lambda\alpha}{2}u^2\right| \leq \frac{\lambda\bar{a}^2}{2\underline{b}^2}\alpha = \frac{\lambda\bar{a}^2}{2\underline{b}^2}\frac{\lambda}{\tau}\left(2CR + \frac{1}{2}\right) \leq \tau. \quad (3.33)$$

Therefore, from (3.31), (3.32), and (3.33), we deduce

$$\int_\Omega\left[\tau(u-v)^2 + \tau(r-s)^2\right]$$
$$\leq \int_\Omega (u-v)(r-s)\left(2bs - \frac{\lambda\alpha}{2}\left(\frac{1}{\alpha}-s\right)^2 - \frac{\lambda\alpha}{2}u^2\right)$$
$$\leq 2\tau\int_\Omega |u-v|\,|r-s|.$$

Hence, we deduce that $r \equiv s$ and so $u \equiv v$ in Ω. ∎

This theorem implies the following

COROLLARY 3.9 [3]. *Let us consider the problem* $(P_{\Omega, a, b, \lambda})$. *Assume that the domain* Ω *and the functions* a *and* b *are fixed satisfying hypothesis* [H1] *and* $\sigma_1(-a) < 0$. *Then there exists* $\Lambda_2 \equiv \Lambda_2(\Omega, a, b) > 0$ *such that for*

$$\lambda \leq \Lambda_2$$

the problem $(P_{\Omega, a, b, \lambda})$ *has a unique optimal control.*

and

COROLLARY 3.10 [4]. *Suppose* [H1], $\sigma_1(-a) < 0$, *and*

$$\overline{b} \leq C\underline{b} \qquad \text{for some } 1 \leq C < 2. \tag{3.34}$$

Then problem $(P_{\Omega, a, b, \lambda})$ *admits an unique optimal control, provided that* $\underline{b}$ *is large enough.*

It is possible to approximate the optimal control by using a monotone scheme according to ideas contained in [9, Chap. V]. To do it, observe that the formula (2.12) expresses any optimal control in terms of the solution of the optimality system (2.13) and (2.14). Thus, we can try to approach the unique optimal control (Theorem 1.1) by a sub−super solution method if an appropriate couple of sub−super solutions is used. The main difficulty in applying the sub−super solution method is the change of monotonicity of the optimality system. For the sake of simplicity, we define the functions $B, C, D\colon \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\begin{aligned} B(x, u, p) &= u\left[a - \left[b + \frac{\lambda}{2}(1 - p)\right]u\right], \\ C(x, u, p) &= p(a - 2bu), \\ D(x, u, p) &= \frac{\lambda}{2}u(1 - p)^2. \end{aligned}$$

One may choose a positive constant G such that

- $B(x, u, p) + Gu$ is increasing in u in the interval $[0, \lambda/\underline{b}]$, for any $x \in \Omega$ and $p \in [0, (\lambda/\underline{b})R]$, and increasing in p in the interval $[0, (\lambda/\underline{b})R]$, for any $x \in \Omega$ and $u \in [0, \lambda/\underline{b}]$ $(\nearrow u, \nearrow p)$.
- $C(x, u, p) + (G/2)p$ is decreasing in u in the interval $[0, \lambda/\underline{b}]$, for any $x \in \Omega$ and $p \in [0, (\lambda/\underline{b})R]$, and increasing in p in the interval $[0, (\lambda/\underline{b})R]$, for any $x \in \Omega$ and $u \in [0, \lambda/\underline{b}]$ $(\searrow u, \nearrow p)$.
- $D(x, u, p) + (G/2)p$ is increasing in u in the interval $[0, \lambda/\underline{b}]$ for any $x \in \Omega$ and $p \in [0, (\lambda/\underline{b})R]$, and increasing in p in the interval $[0, (\lambda/\underline{b})R]$, for any $x \in \Omega$ and $u \in [0, \lambda/\underline{b}]$ $(\nearrow u, \nearrow p)$.

Now, consider $u_1 \equiv w$, $u^1 \equiv \bar{a}/\underline{b}$, $p_1 \equiv 0$, and $p^1 \equiv (\lambda/\underline{b})R$, with w. $\mathscr{Q}$ defined in (2.16) and (2.20), respectively, and the sequences $\{u_n\}$, $\{u^n\}$, $\{p_n\}$, and $\{p^n\}$ as the unique solutions of

$$\begin{aligned}
-\Delta u_{n+1} + Mu_{n+1} &= B(x, u_n, p_n) + Mu_n, && \text{in } \Omega,\\
u_{n+1} &= 0, && \text{on } \partial\Omega,\\
-\Delta u^{n+1} + Mu^{n+1} &= B(x, u^n, p^n) + Mu^n, && \text{in } \Omega,\\
u^{n+1} &= 0, && \text{on } \partial\Omega,\\
-\Delta p_{n+1} + Mp_{n+1} &= C(x, u^n, p_n)\\
&\quad + \frac{M}{2} p_n + D(x, u_n, p_n) + \frac{M}{2} p_n, && \text{in } \Omega,\\
p_{n+1} &= 0, && \text{on } \partial\Omega,\\
-\Delta p^{n+1} + Mp^{n+1} &= C(x, u_n, p^n)\\
&\quad + \frac{M}{2} p^n + D(x, u^n, p^n) + \frac{M}{2} p^n, && \text{in } \Omega,\\
p^{n+1} &= 0, && \text{on } \partial\Omega.
\end{aligned}$$

Taking into account the monotonicity properties of functions B, C, D, and standard arguments of sub−super solution method, we have that $\{u_n\}$, $\{u^n\}$, $\{p_n\}$, and $\{p^n\}$ verify

1.
$$\begin{aligned}
&u_1 \le u_2 \le \cdots \le u_n \le u^n \le u^{n-1} \le \cdots \le u^1, && \text{in } \Omega,\\
&p_1 \le p_2 \le \cdots \le p_n \le p^n \le p^{n-1} \le \cdots \le p^1, && \text{in } \Omega,
\end{aligned}$$

and

$$u_n \nearrow u_*, u^n \searrow u^*, p_n \nearrow p_*, p^n \searrow p^*, \qquad \text{pointwise in } \Omega.$$

2. $u_*, u^*, p_*, p^* \in W_0^{2,q}(\Omega)$, $\forall q \in (1, \infty)$. Moreover, these functions satisfy

$$\begin{aligned}
-\Delta u_* &= B(x, u_*, p_*), && \text{in } \Omega,\\
-\Delta u^* &= B(x, u^*, p^*), && \text{in } \Omega,\\
-\Delta p_* &= C(x, u^*, p_*) + D(x, u_*, p_*), && \text{in } \Omega,\\
-\Delta p^* &= C(x, u_*, p^*) + D(x, u^*, p^*), && \text{in } \Omega,\\
u_* &= u^* = p_* = p^* = 0, && \text{on } \partial\Omega
\end{aligned} \tag{3.35}$$

with conditions

$$w \le u_*, u^* \le \frac{\bar{a}}{\underline{b}}, \qquad 0 \le p_*, p^* \le \frac{\lambda R}{\underline{b}}. \tag{3.36}$$

3. If (u, p) is another solution of (2.13) with the property

$$u_1 \le u \le u^1,\ p_1 \le p \le p^1,$$

then

$$u_* \le u \le u^*,\ p_* \le p \le p^*.$$

Note that if system (3.35) and (3.36) has a unique solution, (u_*, u^*, p_*, p^*), then (u^*, u_*, p^*, p_*) is another solution of (3.35) and (3.36), and consequently $u_* = u^*$ and $p_* = p^*$ is a solution of system (2.13) and (2.14). This will give a (theoretical) approximation to the optimal control.

The next theorem shows that under more restrictions on quantity $\underline{b}/\lambda$ system (3.35) and (3.36) has a unique solution. The ideas to prove it are similar to the ones in Theorem 1.1.

THEOREM 3.11 (approx. to the optimal control). *Assume* [H1], [H2], *and* $\sigma_1(-a) < 0$. *Take* $\beta = \max\{2CR/\tau, 1/\tau\}$. *If*

$$\frac{\underline{b}^2}{\lambda^2} \ge \max\left\{\Gamma, \frac{\beta \bar{a}^2}{\tau}\right\}, \tag{3.37}$$

then, *system* (3.35) *and* (3.36) *has a unique solution*. *Note that* Γ, τ, *and* R *are considered as in the proof of Theorem* 1.1.

Proof. The main argument is the following: we will assume the existence of two solutions of system (3.38) satisfying (3.39), and then we will show that, udner the requirements of the theorem, they are the same.

Consider (u, v, p, q) and (U, V, P, Q) as two solutions of system (3.35) and (3.36) or equivalently $(u, v, s, t) \equiv (u, v, p/\lambda\beta, q/\lambda\beta)$ and $(U, V, S, T) \equiv (U, V, P/\lambda\beta, Q/\lambda\beta)$ as solutions of system

$$\begin{aligned}
-\Delta u - au + bu^2 + \frac{\lambda}{2}u^2 - \frac{\lambda^2\beta}{2}su^2 &= 0, && \text{in } \Omega,\\
-\Delta v - av + bv^2 + \frac{\lambda}{2}v^2 - \frac{\lambda^2\beta}{2}tv^2 &= 0, && \text{in } \Omega,\\
-\Delta s - as + 2bvs - \frac{\lambda^2\beta}{2}u\left(\frac{1}{\lambda\beta} - s\right)^2 &= 0, && \text{in } \Omega,\\
-\Delta t - at + 2but - \frac{\lambda^2\beta}{2}v\left(\frac{1}{\lambda\beta} - t\right)^2 &= 0, && \text{in } \Omega,\\
u = v = r = s &= 0, && \text{on } \partial\Omega,
\end{aligned}$$

satisfying

$$0 \le s(x), t(x) \le \frac{\lambda R}{\lambda \beta \underline{b}} \left(\le \frac{1}{\lambda \beta} \right); w(x) \le u(x), v(x) \le \frac{\bar{a}}{\underline{b}}, \qquad \text{a.e. in } \Omega. \tag{3.39}$$

Now, taking into account that (U, V, S, T) is another solution of (3.38), by subtracting, multiplying by $(u - U)$, and integrating on Ω in the first equation of both systems, we have

$$\begin{aligned} 0 = \int_\Omega \Bigg[& |\nabla(u - U)|^2 - a(u - U)^2 + b(u + U)(u - U)^2 \\ & + \frac{\lambda}{2}(u + U)(u - U)^2(1 - \lambda\beta S) - \frac{\lambda^2\beta}{2}u^2(u - U)(s - S) \Bigg]. \end{aligned} \tag{3.40}$$

We use the same process with the third equation

$$\begin{aligned} 0 = \int_\Omega \Bigg[& |\nabla(s - S)|^2 - a(s - S)^2 + 2bv(s - S)^2 + 2bS(v - V)(s - S) \\ & + \frac{\lambda}{2}u(s - S)^2[2 - \lambda\beta(s + S)] - \frac{\lambda^2\beta}{2}(u - U)(s - S)\left(\frac{1}{\lambda\beta} - s\right)^2 \Bigg]. \end{aligned} \tag{3.41}$$

Following the same arguments with the second and fourth equations of system (3.38), by adding the corresponding equalities (3.39) and the properties of τ (recall also (2.5)), we get

$$\begin{aligned} 0 \ge \int_\Omega \Bigg\{ & \tau\left[(u - U)^2 + (v - V)^2 + (s - S)^2 + (t - T)^2\right] \\ & - \frac{\lambda^2\beta}{2}\left[u^2 + \left(\frac{1}{\lambda\beta} - s\right)^2\right](u - U)(s - S) \\ & - \frac{\lambda^2\beta}{2}\left[v^2 + \left(\frac{1}{\lambda\beta} - T\right)^2\right](v - V)(t - T) \\ & + 2bS(v - V)(s - S) + 2bT(u - U)(t - T) \Bigg\}. \end{aligned} \tag{3.42}$$

Moreover, if $s \neq S$ or $t \neq T$ in a subset of Ω with positive measure, then, similarly to the proof of Theorem 1.1, the previous inequality is strict. From (3.39) and the choice of β it follows that

$$|2bS| \leq \frac{2CR}{\beta} \leq \tau \qquad \text{and accordingly } |2bT| \leq \tau. \tag{3.43}$$

On the other hand, as $\beta \geq (1/\tau)$ and $(b^2/\lambda^2) \geq (\beta\bar{a}^2/\tau)$ (see (3.37)), one infers that $(\lambda^2\beta/2)(1/(\lambda\beta)^2) \leq (\tau/2)$ and $(\lambda^2\beta/2)(\bar{a}^2/\underline{b}^2) \leq (\tau/2)$, respectively. Consequently, by virtue of (3.39),

$$\left| -\frac{\lambda^2\beta}{2}\left[u^2 + \left(\frac{1}{\lambda\beta} - S\right)^2\right]\right| \leq \frac{\tau}{2} + \frac{\tau}{2} = \tau, \tag{3.44}$$

and

$$\left| -\frac{\lambda^2\beta}{2}\left[v^2 + \left(\frac{1}{\lambda\beta} - T\right)^2\right]\right| \leq \frac{\tau}{2} + \frac{\tau}{2} = \tau. \tag{3.45}$$

Thus, the right side of (3.42) is nonnegative. Hence, $s = S$ and $t = T$ in Ω and therefore $u = U$ and $v = V$ in Ω. ∎

The previous theorem allow us to approximate the unique optimal control by an iterative scheme provided that $\frac{b}{\lambda}$ is large enough.

FINAL REMARKS

1. Note that Theorems 1.1 and 3.11 are valid in the case in which we consider the logistic equation (1.1) with Neumann boundary conditions. Tools and techniques used in this paper (especially Lemmas 2.7 and 2.8) can be used to prove them. In this case, the elliptic equations and systems which appear in this paper should be changed by the corresponding ones with Neumann boundary conditions.

2. Also, the space of controls may be changed by taking $C_\delta = \{g \in L^\infty(\Omega) : 0 \leq g \leq \delta\}$ instead of $L^\infty_+(\Omega)$. If, in addition, one considers $\underline{b}/\lambda > \bar{a}/\delta$ (recall that, by virtue of (2.10), the optimal control satisfies $0 \leq f \leq \lambda\bar{a}/\underline{b}$ and then belongs to C_δ) then one obtains again that Theorems 1.1 and 3.11 hold. Hence, we answer positively about the uniqueness and approximation to the optimal control proposed in [10, Remark 4.1].

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